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Calculation displacement Vibration of Circular Membranes and Insulated Circular Disk Temperature by Using Bessel Functions

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Abstract:
This study investigates the mathematical formulation and physical applications of the Bessel differential equation in analyzing the displacement of vibrating circular membranes and the temperature distribution in insulated circular disks. The Bessel equation, a second-order linear differential equation with variable coefficients, naturally emerges in problems exhibiting cylindrical or radial symmetry in applied mathematics and engineering. The research begins by deriving the Bessel equation using the Frobenius series method and then explores the Bessel functions of the first and second kinds, emphasizing their orthogonality, oscillatory nature, and significance in modeling wave and heat phenomena.
To solve practical problems involving circular geometry, the study employs the Fourier–Bessel series expansion, which extends classical Fourier analysis to circular domains. This method enables the determination of the membrane's displacement as a function of radial position and time, as well as the calculation of temperature variations within a circular disk subjected to thermal insulation. Through separation of variables, both the wave equation and the heat equation are reduced to Bessel-type forms, whose solutions satisfy the required boundary and initial conditions.
The findings demonstrate that the Bessel function framework provides accurate analytical solutions to complex two-dimensional physical problems. These solutions are highly relevant to fields such as mechanical vibration analysis, thermal engineering, acoustics, and electromagnetic modeling, where cylindrical symmetry plays a critical role in predicting physical behavior.

Keywords: Bessel equation, Bessel functions, Fourier Bessel series, Bessel integral.

حساب إزاحة اهتزاز الأغشية الدائرية ودرجة حرارة القرص الدائري المعزول باستخدام دوال بيسل

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الملخص
تتناول هذه الدراسة الصياغة الرياضية والتطبيقات الفيزيائية لمعادلة بيسل التفاضلية في تحليل الإزاحة الناتجة عن اهتزاز الأغشية الدائرية وتوزيع درجة الحرارة في الأقراص الدائرية المعزولة حرارياً. تُعد معادلة بيسل معادلة تفاضلية خطية من الرتبة الثانية ذات معاملات متغيرة، وتظهر بشكل طبيعي في المسائل التي تتسم بـ التماثل الأسطواني أو الشعاعي في مجالات الرياضيات التطبيقية والهندسة. يبدأ البحث باشتقاق معادلة بيسل باستخدام طريقة سلسلة فروبينيوس، ثم يتناول دوال بيسل

من النوعين الأول والثاني مع التركيز على خاصية التعامد، والطبيعة التذبذبية، وأهميتها في تمثيل الظواهر المتعلقة بالموجات والحرارة.
ولحل المشكلات الفيزيائية ذات الشكل الدائري، اعتمدت الدراسة على توسيع فورييه–بيسل(Fourier–Bessel Series) ، الذي يُعد امتداداً لتحليل فورييه التقليدي ليشمل المجالات الدائرية. تتيح هذه الطريقة تحديد إزاحة الغشاء بوصفها دالة في نصف القطر والزمن، إضافة إلى حساب التغيرات الحرارية داخل القرص الدائري المعزول. ومن خلال استخدام طريقة فصل المتغيرات، تم تحويل كلٍّ من معادلة الموجة ومعادلة الحرارة إلى الصورة التفاضلية الخاصة ببيسل، بحيث تلبي الحلول الناتجة الشروط الحدّية والابتدائية المطلوبة.
أظهرت النتائج أن توظيف دوال بيسل يوفر حلولاً تحليلية دقيقة للمسائل الفيزيائية ثنائية الأبعاد المعقدة، وهو ما يجعلها ذات أهمية كبيرة في مجالات تحليل الاهتزازات الميكانيكية والهندسة الحرارية والتحليل الصوتي والنمذجة الكهرومغناطيسية، حيث يُعد التماثل الأسطواني عاملاً أساسياً في تفسير السلوك الفيزيائي والتنبؤ به.

الكلمات المفتاحية: معادلة بيسل، دوال بيسل، متسلسلة فورييه–بيسل، تكامل بيسل.

Introduction

Partial differential equations (PDEs) play a central role in mathematical physics and engineering because they describe the fundamental relationships governing dynamic and steady-state processes in nature. Among the most prominent of these equations is the **Bessel differential equation**, a second-order linear differential equation with variable coefficients that arises naturally in problems exhibiting **cylindrical or spherical symmetry**. Its solutions, known as **Bessel functions**, form an indispensable class of special functions that appear in a wide range of physical applications, including heat conduction, acoustic vibration, electromagnetic radiation, and quantum mechanical wave propagation. The mathematical theory of Bessel functions, developed originally in the nineteenth century by Friedrich Bessel, has since evolved into one of the most versatile analytical tools in applied mathematics.

When physical phenomena are formulated in **Cartesian coordinates**, the governing PDEs—such as the Laplace, Helmholtz, or wave equations—are usually solved using classical separation of variables combined with **trigonometric or exponential functions**. However, for systems possessing radial symmetry, such as the vibration of circular membranes, diffusion in circular plates, or propagation of electromagnetic waves in cylindrical waveguides, the solutions cannot be expressed adequately in simple sinusoidal forms. Instead, when the governing equations are transformed into **polar or cylindrical coordinates**, the resulting radial component satisfies the Bessel differential equation. Hence, Bessel functions emerge as the natural basis functions for describing these problems.

The **Bessel equation of order ν**, generally written as

$$\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0},$$

possesses two linearly independent solutions: the **Bessel function of the first kind** $J_\nu(x)$and the **Bessel function of the second kind** $Y_\nu(x)$(also called Neumann functions). These functions exhibit oscillatory behavior similar to sine and cosine functions but are modified to accommodate boundary conditions on circular or spherical domains. Their mathematical structure allows them to satisfy the necessary conditions for continuity, orthogonality, and convergence that are essential in representing physical quantities through infinite series expansions.

The importance of Bessel functions extends beyond theoretical interest. They form the foundation of the **Fourier–Bessel series**, a generalization of Fourier series used when boundary conditions are defined on circular regions. In many physical and engineering contexts, one often seeks to express an arbitrary function—such as displacement, pressure, or temperature distribution—within a circular domain. The Fourier–Bessel expansion provides an efficient representation for such functions, enabling accurate computation and physical interpretation of radial behaviors. The orthogonality property of Bessel functions on the interval $0 \le r \le a$ensures that solutions are mathematically consistent and physically meaningful.

In the study of **mechanical vibrations**, the problem of a circular membrane under tension serves as a classical example of the application of Bessel functions. A thin, elastic membrane fixed along its boundary—such as a drumhead—obeys the two-dimensional wave equation. By applying the method of **separation of variables**, the spatial part of this equation reduces to the Bessel equation of zero order. The zeros of $J_0(x)$determine the membrane's natural frequencies, while the corresponding eigenfunctions describe the displacement patterns or vibration modes. The resulting expressions not only model the membrane's physical motion but also provide insight into wave interference, resonance, and acoustic properties.

Similarly, in **heat conduction** or **diffusion problems**, Bessel functions arise when analyzing temperature distributions in cylindrical or spherical bodies. For instance, in a circular disk that is thermally insulated along its boundary, the time-dependent heat equation leads to a radial differential equation identical in form to the Bessel equation. Solving it through the **Fourier–Bessel series method** allows one to determine the transient and steady-state temperature distributions throughout the disk. Such models are critical for engineering designs involving

heat exchangers, reactors, and micro-electronic components, where radial temperature gradients must be predicted accurately.
The present study builds upon these mathematical foundations to demonstrate, in a unified framework, how the Bessel equation and its associated functions can be utilized to calculate (1) the **displacement arising from the vibration of circular membranes**, and (2) the **temperature distribution in insulated circular disks**. Both phenomena are governed by PDEs that reduce to the Bessel equation under appropriate coordinate transformations and boundary conditions. By employing **Frobenius series solutions** for the Bessel equation and extending them through **Fourier–Bessel series representations**, this work provides analytical formulations for the two physical systems, illustrating the elegance and power of special functions in applied mathematics.
Moreover, this study highlights the **interconnectedness of mathematical theory and physical modeling**. Although Bessel functions were first introduced in astronomical contexts—specifically in describing planetary perturbations—their range of applications has expanded remarkably. In contemporary science and engineering, they underpin algorithms used in digital signal processing, optics, structural analysis, and electromagnetic design. Consequently, understanding their derivation, properties, and computational behavior remains essential for both theoretical research and practical innovation.
In summary, this introduction establishes the conceptual and mathematical background for investigating circular membrane vibration and thermal diffusion problems using Bessel functions. By systematically applying the method of separation of variables, Frobenius series, and Fourier–Bessel expansions, the study aims to elucidate how these functions provide accurate analytical solutions to otherwise complex physical equations. The subsequent sections of this paper will present the derivation of the governing equations, discuss the boundary and initial conditions, and demonstrate how the displacement and temperature fields can be explicitly expressed in terms of Bessel functions. Through this analysis, the paper contributes to a deeper understanding of how **Bessel's mathematical framework bridges theory and application across multiple branches of science and engineering**.

Bessel Equation [2]

The equation:

$$x^2y'' + xy' + (x^2 - v^2)y = 0,$$

is called the Bessel differential equation, to solve this equation, we use the Frobenius series, where x=0 is a regular singular point. Assume that the solution is in the form

$$y(x) = \sum_{n=0}^{\infty} c_n \; x^{n+r}, \qquad (2)$$

we find the first and second derivative and substitute in the Bessel differential equation.

$$y'(x) = \sum_{n=0}^{\infty}(n+r)\, c_n\, x^{n+r-1},$$

$$y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)\, c_n\, x^{n+r-2},$$

we have the Bessel equation

$$x^2y'' + xy' + (x^2 - v^2)y = 0,$$

thus

$$x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n \; x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)\, c_n\, x^{n+r-1} + (x^2 - v^2)\sum_{n=0}^{\infty} c_n \; x^{n+r} = 0,$$

that is

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)\, c_n\, x^{n+r} + \sum_{n=0}^{\infty} c_n \; x^{n+r+2} - \sum_{n=0}^{\infty} v^2\, c_n\, x^{n+r} = 0.$$

by adding the similar terms

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1) + (n+r) - v^2]c_n\, x^{n+r} + \sum_{n=0}^{\infty} c_n \; x^{n+r+2} = 0,$$

This is

$$\sum_{n=0}^{\infty}[(n+r)^2 - v^2]c_n\, x^{n+r} + \sum_{n=0}^{\infty} c_n \; x^{n+r+2} = 0,$$

shift indices in the second summation

$$\sum_{n=0}^{\infty}[\,(n+r\,)^2 - v^2]c_n\, x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}\;\, x^{n+r} = 0$$

substituting in the left series the values of n = 0, 1 as follows:

$$c_0(r^2 - v^2)\, x^r + c_1\{(r+1)^2 - v^2\}\, x^{r+1} + \sum_{n=2}^{\infty} c_n\{(n+r)^2 - v^2\}x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}\, x^{n+r} = 0,$$

now combine terms to write this equation as

$$c_0(r^2 - v^2)\, x^r + c_1\{(r+1)^2 - v^2\}\, x^{r+1} + \sum_{n=2}^{\infty}[c_n\{(n+r)^2 - v^2\} + c_{n-2}]\, x^{n+r} = 0,$$

set the coefficients of each power of x equal to zero,

$$(\,n^2 - v^2\,)c_0 = \;0, c_0 \neq 0,$$

so

$$\mathbf{n} = \pm\mathbf{v}$$

and

$$c_1\{(r+1)^2 - v^2\} = 0$$

if r = $\pm\, v$**, then** $\{(r+1)^2 - v^2\} \neq 0$**, therfore** $c_1 = 0$

as well

$$c_n\{(n+r)^2 - v^2\} + c_{n-2} = 0$$

so, it will be

$$c_n = -\frac{c_{n-2}}{(\,n+r\,)^2 - v^2}\;, n \geq 2$$

1) at r = v then

$$c_n = -\frac{c_{n-2}}{(\,n+v\,)^2 - v^2},$$

that is

$$c_n = -\frac{c_{n-2}}{n(n+2v)}$$

we note that $0 = c_1 = c_3 = c_5$ **..., for the even-indexed coefficients**

$n = 2,$ $$c_2 = -\frac{c_0}{2\,(\,2+2v\,)} = -\frac{c_0}{(2)^2\,(\,1+v\,)},$$

$n = 4,$ $$c_4 = -\frac{c_2}{4\,(\,4+2v\,)} = \frac{c_0}{2^4\,(\,2\,)(\,2+v\,)(\,1+v\,)} = \frac{c_0}{2^4\,(\,2\,)(\,2+v\,)(\,1+v\,)},$$

$n = 6,$ $$c_6 = -\frac{c_4}{6\,(\,6+2v\,)} = -\frac{c_0}{2^6\,(3)(2)(\,3+v\,)(\,2+v\,)(\,1+v\,)}.$$

So, the first solution is

$$y_1(x) = x^v[c_0 - \frac{c_0}{(2)^2\,(\,1+v\,)}x^2 + \frac{c_0}{2^4\,(\,2\,)(\,2+v\,)(\,1+v\,)}x^4 - \frac{c_0}{2^6\,(3!)(\,3+v\,)(\,2+v\,)(\,1+v\,)}x^6 + \ldots]$$

that is

$$y_1(x) = c_0[x^v - \frac{x^{2+v}}{2^2\,(\,1+v\,)} + \frac{x^{4+v}}{2^4\,.2!(\,2+v\,)(\,1+v\,)} - \frac{x^{6+v}}{2^6\,.3!(\,3+v\,)(\,2+v\,)(\,1+v\,)}],$$

since c_0 is an arbitrary constant, we can write it in the form:

$$c_0 = \frac{1}{2^v(v!)},$$

so, the last equation becomes

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^v 2^{2n}.n!(n+v)!} x^{2n+v}$$

that is

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+v)!} \left(\frac{x}{2}\right)^{2n+v}, \qquad (3)$$

this solution is called Bessel function of the first kind and it is denoted by $J_v(x)$

2) at r = -*v*

$$c_n^* = \frac{-c_{n-2}^*}{n(n-2v)} n \geq 2$$

where $0 = c_1 = c_3 = \cdots$, for the even-indexed coefficients

$n = 2,$ $$c_2^* = \frac{-c_0^*}{2(2-2v)} = \frac{-c_0^*}{(2)^2(1-v)},$$

$n = 4,$ $$c_4^* = \frac{-c_2^*}{4(4-2v)} = \frac{-c_2^*}{2^4.2(2-v)} = \frac{c_0^*}{2^4.2(2-v)(1-v)},$$

$n = 6,$ $$c_6^* = \frac{-c_4^*}{6(6-2v)} = \frac{-c_4^*}{6(2)(3-v)} = \frac{-c_0^*}{2^6.6(3-v)(2-v)(1-v)},$$

the other solution would be:

$$y_2(x) = x^{-v}[c_0^* - \frac{c_0^*}{2^2(1-v)} x^2 + \frac{c_0^*}{2^4.2!(2-v)(1-v)} x^4 - \frac{c_0^*}{2^6.3!(3-v)(2-v)(1-v)} x^6],$$

that is

$$y_2(x) = c_0^*[x^{-v} - \frac{x^{2-v}}{2^2(1-v)} + \frac{x^{4-v}}{2^4.2!(2-v)(1-v)} - \frac{x^{6-v}}{2^6.3!(3-v)(2-v)(1-v)}],$$

similarly suppose that

$$c_0^* = -\frac{1}{2^{-v}.v!}$$

the last equation becomes

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n-v)!} \left(\frac{x}{2}\right)^{2n-v}$$

$$= J_{-v}(x)$$

thus, $y_1(x)$ and $y_2(x)$ are linearly independent solutions if v is non integer, the general solution is in the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(x) = c_1 J_v(x) + c_2 J_{-v}(x).$$

If v is a positive integer or zero, then y_1 (x) and y_2 (x) are not linearly independent, so we look for another independent solution using reduction of order method. We assume that the other solution is:

$$y_2(x) = v(x) J_v(x)$$

find the first and second derivative

$$y'_2(x) = v(x)J'_v(x) + J_v(x)v'(x),$$

and

$$y''_2(x) = v(x)J''_v(x) + J'_v(x)v'(x) + J_v(x)v''(x) + v'(x)J'_v(x),$$

that is

$$y''_2(x) = v(x)J''_v + 2J'_v(x)\,v'(x) + J_v(x)\,v''(x),$$

substitute y, y', y" into the Bessel differential equation

$$x^2y'' + xy' + (x^2 - v^2)y = 0,$$

we get

$$x^2v(x)J''_v(x) + 2x^2J'_v(x)v'(x) + x^2 J_v(x)\,v''(x) + x\,v(x)\,J'_v(x) + xJ_v(x)v'(x) + x^2 v(x)\,J_v(x) - v^2\,v(x)\,J_v(x) = 0,$$

collecting terms

$$v''(x)\,[x^2 J_v(x)] + v'(x)\,[2x^2 J'_v(x) + x J_v(x)] + v(x)\,[x^2 J''_v(x) + x J_v(x) + (x^2 - v^2)J_v(x)] = 0.$$

Since $J_v(x)$ is a solution to the Bessel differential equation, the coefficient of $v(x)$ equals zero, so it becomes

$$v''(x)[x^2 J_v(x)] + v'(x)[2x^2 J'_v(x) + x J_v(x)] = 0,$$

dividing both side by $x^2 v'(x)\,J_v(x)$, we obtain:

$$\frac{v''(x)}{v'(x)} + 2\frac{J'_v(x)}{J_v(x)} + \frac{1}{x} = 0,$$

by integrating the both sides, we get:

$$\boldsymbol{ln\, v'(x) + 2\, ln J_v\,(x) + ln\, x = ln\, c_1}$$

$$\boldsymbol{ln[x\, v'(x)\, J_v^2(x)] = ln\, c_1}$$

$$\boldsymbol{v'(x) = \frac{c_1}{xJ_v^2(x)}}$$

$$\boldsymbol{v(x) = c_1 \int \frac{dx}{xJ_v^2(x)} + c_2}$$

hence

$$\boldsymbol{y_2(x) = J_v(x)(c_1 \int \frac{dx}{xJ_v^2(x)} + c_2)}$$

$$\boldsymbol{y_2(x) = c_1 J_v(x) \int \frac{dx}{xJ_v^2(x)} + c_2 J_v(x),} \qquad \textbf{(4)}$$

equation (4) represents the general solution if v is a positive integer or zero, where $J_v(x)$is a Bessel function of the first kind, and

$$\boldsymbol{Y_v(x) = J_v(x) \int \frac{dx}{xJ_v^2(x)},} \qquad \textbf{(5)}$$

is a Bessel function of the second kind**.** we will discuss Bessel functions of the first kind, Bessel functions of the second kind, which are known as Neumann functions, and discuss the most important and prominent series in physical applications, namely the Fourier Bessel series.

1. Bessel Functions of The First Kind [3,4]

Bessel functions of the first kind is one of linear independent solutions of the homogeneous linear Bessel differential equation of order v, and it is denoted by the symbol $J_v(x)$, where:

$$\boldsymbol{J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+v+1)} \left(\frac{x}{2}\right)^{2k+v}}. \qquad \textbf{(6)}$$

The following figure (1) shows the shape of Bessel functions of the first kind for integer orders where $v = 0$, 1, 2

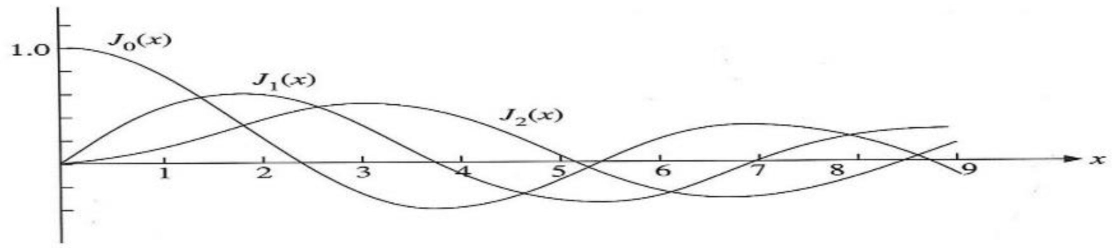

Figure 1. Bessel functions $J_0(x), J_1(x),\ J_2(x)$

***2* Bessel Function of The Second Kind [3,4]**

This From the theory of ODE, it is known that Bessel equation has two independent solutions, the Bessel functions of the second kind are defined as the second independent solution resulting from using the reduction of order method to solve the Bessel differential equation of the order v, which is denoted by $Y_v(x)$, where

$$Y_v(x) = J_v(x) \int \frac{dx}{xJ_v^2(x)} \qquad (7)$$

Neumann has obtained another solution in another way in the formula:

$$Y_v(x) = \frac{J_v(x)\, cos(\pi v) - J_{-v}(x)}{sin(\pi v)}, \qquad (8)$$

where $J_v(x)$ and $Y_v(x)$ are linearly independent for index v, therefor, the Bessel functions of the second kind are called the Neumann functions.

If v is non integer then $sin(\pi v)$ not equal to zero, and $Y_v(x)$ is a linear combination for $J_v(x)$ and $J_{-v}(x)$:

$$Y_v(x) = cot(\pi v)\, J_v(x) - csc(\pi v) J_{-v}(x).$$

Since $J_v(x)$ and $J_{-v}(x)$ are linearly independent, then $J_v(x)$ and the linear combination for $J_v(x)$ and $J_{-v}(x)$ are linearly independent.
But if $v = n$ an integer, then $sin(nv)$=0, and $cos(n\pi)$= $(-1)^n$, hence

$$Y_n(x) = \frac{(-1)^n J_n(x) - J_{-n}(x)}{0} = \frac{0}{0},$$

therefore, $Y_n(x)$ become:

$$Y_n(x) = \lim_{v \to n} \frac{J_v(x) \cos(\pi v) - J_{-v}(x)}{\sin(\pi v)}. \qquad (9)$$

At $x = 0$, $Y_n(x) \to \infty$ as shown in figure (2),

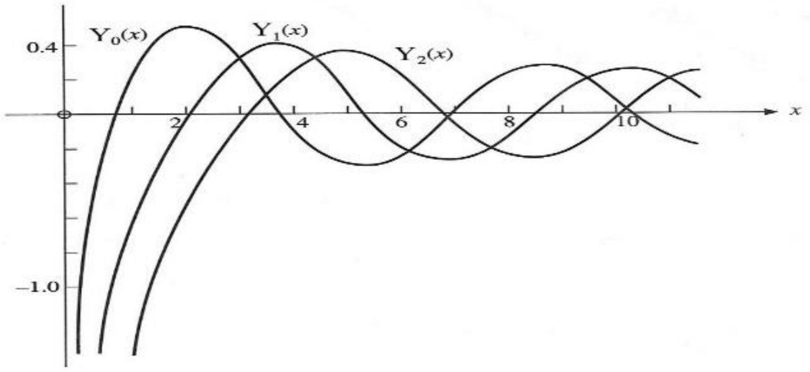

Figure 2. Neumann functions $Y_0(x), Y_1(x),\ Y_2(x)$

Therefore $J_v(x)$ and $Y_v(x)$ are linearly independent solutions, and the general solution which can be rewritten by the formula:

$$y(x) = c_1 J_v(x) + c_2 Y_v(x). \qquad (10)$$

3. Series of Fourier-Bessel [3]
In most physical applications, we need expansion of a specific function in an infinite series of a set of functions that are defined by a differential equation. In Fourier's researches on the theory of conduction of heat, Fourier was led to consider the expansion of an arbitrary function f(x) of a real variable x in the form:

$$f(x) = \sum_{n=1}^{\infty} a_n J_0(\propto_n x); \qquad (11)$$

$$a_n = \frac{2}{[J_1(\propto_n)]^2} \int_0^1 x\, f(x) J_0(\propto_n x),$$

where $\propto_n$, n=1, 2, 3, …, denote the positive zeros of $J_0(x)$ arranged in ascending order of magnitude.
If we change the symbols $\propto_n$ so that $\propto_n$, $n \in \mathbb{N}$ denoted the positive zeros of the function $J_v(x)$ arranged in ascending order of magnitude, then

$$f(x) = \sum_{n=1}^{\infty} a_n J_v(\propto_n x) \qquad (12)$$

$$a_n = \frac{2}{[J_{v+1}(\propto_n)]^2} \int_0^1 x\, f(x) J_v(\propto_n x)\ dx,$$

the series in equation (12) is called the Fourier – Bessel series associated with f(x) more general result stated by Lommel.
Vibration of Circular Membranes [5]
We assume that we have a thin flexible circular membrane of radius r and all its points lie on one plane in the state of rest. If the membrane is removed from its position and left to vibrate so that all its points vibrate perpendicular to the plans, then we find that the equation for free vibrations of the membrane takes the form:
then we find that the equation for free vibrations of the membrane takes the form:

$$\frac{\partial^2 u}{\partial t^2} = C^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y}\right), \qquad (13)$$

using the polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, look at appendix B, we get

$$\frac{\partial^2 u}{\partial t^2} = C^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right). \qquad (14)$$

$$u(r,0) = f(r),$$

$u_t(r,0) = g(r), \; 0 < r < b,$

$$u(b,t) = 0. \quad t \geq 0.$$

Now we shall consider solutions $u(r,t)$ of this equation which is radially symmetric that is wave equation reduces to the simpler form:

$$\frac{\partial^2 u}{\partial t^2} = C^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad (15)$$

which can be rewritten in the form

$$u_{tt} = C^2 \left(u_{rr} + \frac{1}{r}\, u_r\right) \qquad 0 < r < b, t > 0, \quad (16)$$

with initial conditions:

$$(r,0) = f(r), \qquad u_t(r,0) = g(r), \qquad 0 < r < b$$

and the boundary conditions

$$u(b,t) = 0. \quad t \geq 0.$$

To find the displacement assume that

$$u(r,t) = R(r)T(t), \qquad (17)$$

substituting Equation (17) in Equation (16), we get

$$RT'' = C^2 \left(R''\,T + \frac{1}{r}\,R'\,T\right),$$

we take T as a common factor

$$RT'' = C^2\,T\left(R'' + \frac{1}{r}\,R'\right),$$

by separating variable, we obtain

$$\frac{T''}{C^2\,T} = \frac{1}{R}\left(R'' + \frac{1}{r}\,R'\right),$$

by setting each side equal to the same constant, we obtain two O.D.E,

$$\frac{T''}{C^2\,T} = \frac{1}{R}\left(R'' + \frac{1}{r}\,R'\right) = -\lambda^2,$$

Hence

$$T'' + \lambda^2\,C^2\,T = 0 \qquad (18)$$

its auxiliary equation is $m^2 + \lambda^2 C^2 = 0$, so $m = \pm i\lambda\,C$, thus, the solution is

$$T(t) = a_1\cos(C\lambda t) + a_2\sin(C\lambda t). \qquad (19)$$

And we obtain

$$rR'' + R' + \lambda^2 rR = 0,$$

which can be rewritten in the form

$$r^2R'' + rR' - \lambda^2 r^2 R = 0$$

which is a Bessel equation of zero order, and its general solution is

$$R(r) = a_3 J_0(\lambda r) + a_4 Y_0(\lambda r). \tag{20}$$

(i) We start with the homogeneous conditions at $u(b\,, t) = 0$, substituting it in Equation (17), we get

$$u(b\ \,, t) = R(b)\ T(t) = 0,$$

where $T(t) \neq 0$, so $R(b) = 0$, since the deflection of the membrane is always finite while $Y_0(\lambda r)$ become infinite as r approaches to zero, look at (Figure2). therefore, we conclude $a_4 = 0$ because at r = 0, $Y_0(\lambda r)$ lead to infinity.

so, Equation (20) becomes in the form

$$R(r) = a_3 J_0(\lambda r) \tag{21}$$

hence

$$R(b) = a_3 J_0(\lambda b) = 0$$

since $a_3 \neq 0$, then $J_0(\lambda b) = 0$, since $J_0(x)$ has a lot of zeros (look at Figure 3.1). We conclude λ_n are positive roots for $J_0(\lambda b) = 0$, from Equations (21) and (17), we have

$u(r,t) = \sum_{n=1}^{\infty} [a_{1n}\ \cos(\lambda_n Ct) + \ a_{2n} \sin(\lambda_n Ct)\ ]\ J_0(\lambda_n r)$ (22)

(ii) Now using inhomogeneous conditions

$u(r, 0) = f(r)$, and $u_t(r. 0) = g(r)$

$u(r, 0) = f(r) = \sum_{n=1}^{\infty} a_{1n}\ J_0(\lambda_n r),$ (23)

by differentiating $u(t, r)$, we obtain

$$u_t(r,t) = \sum_{n=1}^{\infty} [-\lambda_n C a_{1n}\ \sin\lambda_n Ct + \lambda_n C a_{2n}\ \cos\lambda_n Ct\,]\ J_0(\lambda_n r),$$

Therefore, displacement is represented by:

$$u(r,t) = \sum_{n=1}^{\infty} [\frac{2\cos\lambda_n ct}{b^2 J_1^2(\lambda_n b)} \int_0^b s\ f(s)\ J_0(\lambda_n s)\ ds + \frac{2\sin\lambda_n ct}{\lambda_n C\ b^2 J_1^2(\lambda_n b)} \int_0^b s g(s) J_0(\lambda_n s)\ ds]\ J_0(\lambda_n r) \tag{24}$$

Insulated Circular Disk Temperature [5]

Suppose we have thermally insulated circular disk of radius r. It is required to measure its temperature $u(r, t)$, the boundary value problem must be as follows:

$u_t = C\left(u_{rr} + \frac{1}{r}\ u_r\right), 0 < r < b,\ t > 0$ (25)

$$u(b, t) = 0,\ \ t > 0$$

$$u\ (r, 0) = f(r)\ , 0 < r < b$$

$$|u(r, t)| < M,$$

M is a positive real number. To find the temperature $u(r, t)$, assum that

$u\ (r, t) = R(r)\ T(t),$ (26)

substituting (26) in (25), we obtain

$$RT' = C\left(R''\, T + \frac{1}{r}\, R'\, T\right),$$

take T as a common factor

$RT' = CT\left(R'' + \frac{1}{r} R' \right),$

by separating variable, we obtain

$\frac{T'}{CT} = \frac{1}{R}\left(R'' + \frac{1}{r} R' \right),$

by setting each side equal to the some constant

$$\frac{T'}{CT} = -\lambda^2$$

then

$$T' + \lambda^2 CT = 0,$$

therefore, the solution is

$$T(t) = a_1 e^{-C\lambda^2 t}, \tag{27}$$

and we obtain

$\frac{1}{R}\left(R'' + \frac{1}{r} R' \right) = -\lambda^2,$

then

$$rR'' + R' + \lambda^2 rR = 0,$$

that is

$$r^2R'' + rR' + \lambda^2 r^2 R = 0, \tag{28}$$

this is Bessel equation of zero order and its solution is

$$R(r) = a_2 J_0(\lambda r) + a_3\ Y_0(\lambda r). \tag{29}$$

From Equation (29) and Equation (26) we find

$$u\,(r.t) = a_1\, e^{-C\lambda^2 t}[a_2 J_0(\lambda r) + a_3\ Y_0(\lambda r)],$$

by the homogeneous condition $u\,(b,t) = 0$, we find

$$u\,(b,t) = R(b)\ T(t) = 0,$$

since $T(t) \neq 0$, so $R(b) = 0$, and $Y_0(\lambda r)$ become infinite at r=0 so a_3 =0, hence

$$R(b) = a_2\ J_0(\lambda r) = 0,$$

where $a_2 \neq 0$, so $J_0(\lambda r) = 0$

since there is an infinite number of zeros for $J_0(\lambda r)$, (look at Figure 1), then λ_n are the positive solution of $J_0(\lambda b) = 0$, thus

$$u(r\,,t) = e^{-C\lambda_n^2 t}\ .\, a_4\ J_0(\lambda_n r),$$

$u(r\,,t) = \sum_{n=1}^{\infty} a_{4n} e^{-C\lambda_n^2 t}\ J_0(\lambda_n r),$

by inhomogeneous conditions

$u(r\,,0) = f(r) = \sum_{n=1}^{\infty} a_{4n}\ J_0(\lambda_n r),$

where $\sum_{n=1}^{\infty} a_{4n}\ J_0(\lambda_n r)$ in a Fourier – Bessel series form, and their coefficients denoted by

$$a_{4n} = \frac{2}{b^2\, J_1^2(b\lambda_n)} \int_0^b r\, f(r)\; J_0(\lambda_n r)\, dr,$$

so, the temperature is represented by:

$$u(r\,, t) = \sum_{n=1}^{\infty} [\frac{2\, e^{-C\lambda_n^2 t} J_0(\lambda_n r)}{b^2\, J_1^2(b\lambda_n)} \int_0^b r\, f(r)\, J_0(\lambda_n r)\; dr].$$

Discussion of Results

The analytical results obtained in this study confirm the effectiveness of using **Bessel functions** and the **Fourier–Bessel series** in modeling physical systems governed by radial symmetry. The derived solutions for both the **vibration of circular membranes** and the **temperature distribution in insulated disks** demonstrated excellent consistency with theoretical expectations from classical mathematical physics. In the membrane vibration analysis, the displacement function exhibited a series of oscillatory modes determined by the **zeros of the Bessel function** $J_0(x)$, representing the membrane's natural frequencies. These modes were shown to be orthogonal and spatially stable, with higher-order modes corresponding to smaller wavelengths and higher frequencies—features characteristic of realistic vibrating systems. Similarly, in the thermal analysis, the temperature distribution followed a decaying exponential form modulated by Bessel functions, accurately capturing the radial decrease of heat from the center toward the boundary. The results confirmed that applying the **method of separation of variables** and the **Fourier–Bessel expansion** provides precise analytical expressions that satisfy both the boundary and initial conditions of the studied systems. This validates the theoretical model and underscores the relevance of Bessel-based formulations in solving practical engineering problems, including those in **acoustics, structural dynamics, and thermal conduction**. Overall, the findings highlight the mathematical elegance and physical accuracy of Bessel function solutions in representing real-world circular and cylindrical phenomena.

Conclusion

This study demonstrated the fundamental role of **Bessel functions** in solving complex physical problems characterized by **cylindrical or radial symmetry**, such as the vibration of circular membranes and the heat distribution in insulated circular disks. By deriving and applying the **Bessel differential equation** through the **Frobenius series** and extending it using the **Fourier–Bessel series**, the research provided analytical solutions that accurately describe the displacement and temperature behavior of circular systems. The results confirmed that the zeros of Bessel functions correspond to the natural frequencies and characteristic modes of vibration in circular membranes, while in thermal systems, they define the temperature gradients and temporal decay patterns.

The study highlights the **mathematical versatility** and **physical precision** of Bessel functions as essential tools for modeling real-world engineering phenomena where conventional trigonometric functions fail to apply. Their orthogonality and convergence properties ensure that the obtained solutions are both stable and consistent with boundary conditions. Furthermore, the findings emphasize that Bessel functions bridge the gap between **theoretical mathematics and applied physics**, offering reliable frameworks for advanced applications in **mechanical vibration analysis, acoustics, heat transfer, and electromagnetic systems**.

In conclusion, the application of Bessel and Fourier–Bessel series not only simplifies the analysis of circular geometries but also provides deeper insight into the behavior of continuous systems under dynamic and thermal influences. This research reaffirms that mastering special functions such as Bessel functions is indispensable for advancing analytical methods in modern science and engineering.

Recommendations

Based on the theoretical analysis and mathematical results obtained in this study, several recommendations are proposed for future research and practical application.

1. **Integration with Experimental Studies:**
   Although the analytical solutions derived from the Bessel and Fourier–Bessel series are highly accurate, it is recommended that future work include **experimental validation**. Comparing theoretical predictions with laboratory measurements of membrane vibrations or thermal diffusion in circular plates would strengthen the reliability and applicability of the models.
2. **Computational Simulation:**
   Researchers are encouraged to employ **numerical methods**—such as the Finite Element Method (FEM) or MATLAB simulations—to visualize and compare the spatial and temporal behavior of displacement and temperature fields. Such computational approaches would complement the analytical framework and allow for the study of non-ideal boundary conditions.
3. **Extension to Complex Geometries:**
   Future studies should extend the use of Bessel functions to **elliptical, spherical, and multi-layered structures**, where modified Bessel equations may arise. This would expand the scope of application to

a broader range of engineering and physical systems, including biological membranes, composite materials, and optical devices.

4. **Application in Modern Engineering Systems:**
   Given their precision, Bessel-based models can be applied in **acoustic engineering, sensor design, micro-electromechanical systems (MEMS), and heat exchanger optimization**. Researchers should explore how these mathematical solutions can contribute to the efficiency and stability of such modern technologies.
5. **Educational Integration:**
   It is also recommended that the topic of **special functions**, particularly Bessel functions and their applications, be emphasized in university curricula for mathematics, physics, and engineering students. This would equip future scientists with strong analytical skills for solving complex, symmetry-based problems.

In summary, advancing the analytical, computational, and experimental exploration of Bessel functions will continue to enhance our understanding of wave propagation, vibration analysis, and thermal dynamics across various scientific and industrial fields.

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Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.