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Existence and uniqueness of solutions for a nonlinear differential equation with one-parameter nonlocal condition: A numerical study via MATLAB

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Abstract:
The primary focus of this study is to explore a nonlinear differential equation with one-parameter nonlocal condition. Our analysis establishes rigorous proofs for both the existence and uniqueness of the solutions. Furthermore, we investigate the solution's stability by examining its continuous dependence on the initial data and the parameter. A numerical example is presented with MATLAB tools to validate the theoretical findings and to demonstrate the relationship between the unique solution and governing parameter and initial data.

Keywords: nonlinear differential equation; nonlocal condition; existence and uniqueness of solution; MATLAB.

الوجود والوحدانية لحلول معادلة تفاضلية غير خطية ذات شرط غير محلي أحادي المعلمة: دراسة عددية باستخدام *MATLAB*

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الملخص
يركز هذا البحث بشكل أساسي على دراسة معادلة تفاضلية غير خطية ذات شرط غير محلي ذي بارامیتر واحد. يقدم تحليلنا براهين دقيقة لوجود الحلول و وحدانيتها. علاوة على ذلك، ندرس استقرار الحل من خلال فحص اعتماده المتصل على الشرط الإبتدائي و البارامیتر. نقدم مثالًا عدديًا باستخدام أدوات MATLAB للتحقق من صحة النتائج النظرية ولتوضيح العلاقة بين الحل الوحيد والشرط الإبتدائي والبارامیتر.

الكلمات المفتاحية: المعادلة التفاضلية غير الخطية، الشرط غير المحلي، وجود و وحدانية الحل، ماتلاب.

1. Introduction

Nonlinear differential equations are considered one of the significant topics in mathematical analysis and scientific applications, due to their ability to describe many natural, engineering, and physical phenomena that cannot be adequately represented by simple linear models. The importance of these equations increases when they are associated with nonlocal conditions, since such conditions do not depend only on the value of the solution at a single point, but rather on the behavior of the solution at more than one point or over a certain interval. This makes the mathematical model more suitable for describing problems in which the current state is affected by previous or later values, or by global relations within the considered domain.

Nonlocal problems have received considerable attention in recent studies because of their connection with various applications, such as control theory, population dynamics, heat transfer, and dynamical systems influenced by internal and external factors. Moreover, the presence of a parameter in the nonlocal condition adds an important analytical aspect, as it allows the investigation of the effect of this parameter on the behavior, stability, and sensitivity of the solution with respect to the initial data.

This study focuses on a nonlinear differential problem equipped with a one-parameter nonlocal condition. The main objective is to prove the existence of at least one solution and then to determine sufficient conditions that guarantee the uniqueness of this solution. The study also investigates the continuous dependence of the solution on both the initial data and the parameter, which is an essential property in the analysis of stability. This means that small changes in the given data do not lead to large or irregular changes in the solution.

To achieve these objectives, the study relies on tools from functional analysis and fixed point theory, particularly Schauder's Fixed Point Theorem for proving existence, in addition to the Lipschitz condition for establishing uniqueness. The differential problem is also transformed into an equivalent integral equation, which makes it possible to apply appropriate analytical methods.

In order to support the theoretical findings, a numerical example is presented using MATLAB. The numerical tools are used to illustrate the behavior of the solution and to show the extent to which it is affected by changes in the initial data and the governing parameter. Thus, the study combines both theoretical and numerical approaches, which gives the research a dual scientific value. It does not only prove the mathematical results in an abstract form, but also verifies these results numerically and presents them visually through numerical curves.

Here we focus primarily on the following problem

$$\frac{dx}{dt} = \Phi\big(t, x(t)\big), \quad a.e \quad t \in (0,1] \qquad (1)$$

Subject to the nonlocal condition

$$x(0) + \alpha x(\tau) = x_0 \,, \tau \in (0,1] \,. \qquad (2)$$

The parameter α is a positive real number.

In [7], the authors studied the equation (1) with nonlocal condition

$$\alpha x(\tau) + \beta y(\gamma) = x_0 \,, \tau \in [0, T), \gamma \in (0, T] \,. \qquad (3)$$

Where the parameters $\alpha > 0$ and $\beta > 0$ satisfied the condition $\alpha + \beta \neq 0$ to prove that there exists solutions to the nonlocal problem (1)-(3).

The objective of this paper is to investigate the existence of at least one solution $x \in AC\,[0,1]$ to the problem (1)-(2). Furthermore, we establish the uniqueness of the solution and demonstrate its continuous dependence on the initial data x_0 and the parameter α . Finally, we present a numerical example to demonstrate the solution's sensitivity and its continuous dependence on both initial data and governing parameter.

1. Main Results

Consider the nonlocal problem (1)-(2) under the following hypotheses:

i) The function $\Phi: [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be Lebesgue measurable with respect to t for every $x \in \mathbb{R}$ and continuous with respect to $x \in \mathbb{R}$ for every $t \in [0,1]$.

ii) The function Φ satisfies the following linear growth condition
$|\Phi(t, x)| \leq \mu(t) + a\,|x|$, where $\mu \in L^1[0,1]$ and a is a positive constant

iii) $\int_0^1 \mu(\xi)d\xi \leq \kappa$, where κ is a constant.

iv) $(1+\alpha) \neq a(1+2\alpha)$.

2.1 Integral equation representation

Now we have the following lemma**:**

Lemma 1.

If the solution of the nonlocal problem (1)-(2) exists, then it can expressed by the following integral equation

$$x(t) = \frac{1}{1+\alpha}\left[x_0 - \alpha \int_0^\tau \Phi(\xi, x(\xi))d\xi\right] + \int_0^t \Phi(\xi, x(\xi))d\xi . \quad (4)$$

Proof. Integrating equation (1), we get

$$x(t) = x(0) + \int_0^t \Phi(\xi, x(\xi))d\xi. \quad (5)$$

Put $t = \tau$ in equation (5), then multiply it by the parameter, we obtain

$$\alpha x(\tau) = \alpha\, x(0) + \alpha \int_0^\tau \Phi(\xi, x(\xi))d\xi. \quad (6)$$

Substitute (6) into (2), we get

$$x_0 - x(0) = \alpha\, x(0) + \alpha \int_0^\tau \Phi(\xi, x(\xi))d\xi,$$

which implies to

$$x(0) = \frac{1}{1+\alpha}\left[x_0 - \alpha \int_0^\tau \Phi(\xi, x(\xi))d\xi\right].$$

Therefore, one can obtain the equation (4).

1.2 Existence of solutions

Theorem 1. Suppose that the hypotheses (i) - (iv) hold. Then, the existence of at least one solution $x \in AC[0,1]$ to the nonlocal problem (1)-(2) is guaranteed.

Proof. Let $\Omega_r$be non-empty closed convex subset defined by:

$\Omega_r = \{x \in \mathbb{R}: \|x\| \leq r\} \subset C[0,1]$. The constant r is a positive real number

$$r = \frac{|x_0| + (1+2\alpha)\|\mu\|}{(1+\alpha) - a(1+2\alpha)}.$$

Define the operator H on $C[0,1]$ with $\|x\| = sup_{t\in[0,1]}|x(t)|$ by:

$$Hx(t) = \frac{1}{1+\alpha}\left[x_0 - \alpha \int_0^\tau \Phi(\xi, x(\xi))d\xi\right] + \int_0^t \Phi(\xi, x(\xi))d\xi.$$

In order to confirm the existence of solutions, we seek to locate a fixed point for the operator H, that is, $Hx = x$.

Let $x \in \Omega_r$

$$|Hx(t)| = \left| \frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^{\tau}\Phi\big(\xi, x(\xi)\big)d\xi\right] + \int_0^{t}\Phi\big(\xi, x(\xi)\big)d\xi \right|$$

$$\leq \frac{1}{1+\alpha}\left[|x_0| + \alpha\int_0^{\tau}\left|\Phi\big(\xi, x(\xi)\big)\right|d\xi\right] + \int_0^{t}\left|\Phi\big(\xi, x(\xi)\big)\right|d\xi$$

$$\leq \frac{1}{1+\alpha}\left[|x_0| + \alpha\int_0^{\tau}(|\mu(\xi)| + a\,|x|)\,d\xi\right] + \int_0^{t}(|\mu(\xi)| + a\,|x|)d\xi$$

$$\leq \frac{1}{1+\alpha}\left[|x_0| + \alpha\int_0^{1}(|\mu(\xi)| + a\,|x|)\,d\xi\right] + \int_0^{1}(|\mu(\xi)| + a\,|x|)d\xi$$

$$\leq \frac{1}{1+\alpha}[|x_0| + \alpha\,(\|\mu\| + a\,\|x\|\,)] + (\|\mu\| + a\,\|x\|\,)$$

$$\leq \frac{1}{1+\alpha}[|x_0| + (2\alpha+1)(\|\mu\| + a\,\|x\|\,)] \leq r$$

Consequently, the collection of $\{Hx\}$ is shown to be uniformly bounded over Ω_r and $H\colon \Omega_r \longrightarrow \Omega_r$.

In order to verify the equicontinuous property of the operator H on the subset Ω_r. Let $x \in \Omega_r$ and $0 \leq t_1 < t_2 \leq 1$, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|t_2 - t_1| < \delta$, then

$$|Hx(t_2) - Hx(t_1)|$$

$$= \left| \frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^{\tau}\Phi\big(\xi, x(\xi)\big)d\xi\right] + \int_0^{t_2}\Phi\big(\xi, x(\xi)\big)d\xi - \left(\frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^{\tau}\Phi\big(\xi, x(\xi)\big)d\xi\right] + \int_0^{t_1}\Phi\big(\xi, x(\xi)\big)d\xi\right) \right|$$

$$\leq \int_{t_1}^{t_2}\left|\Phi\big(\xi, x(\xi)\big)\right|d\xi \leq \int_{t_1}^{t_2}(\mu(\xi) + a\,|x|)\,d\xi$$

$$\leq \kappa + a|x|\,|t_2 - t_1| < \varepsilon.$$

Consequently, the equicontinuity of $\{Hx\}$ is established on Ω_r.

Applying the Arzela- Ascoli's Theorem [8], we deduce that $H\colon \Omega_r \to \Omega_r$ is compact.

To ensure the operator H fulfills the continuity on Ω_r, assume that $\{x_n\} \in \Omega_r$ be a convergent sequence such that $x_n \to x$, then

$$Hx_n(t) = \frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^{\tau}\Phi\big(\xi, x_n(\xi)\big)d\xi\right] + \int_0^{t}\Phi\big(\xi, x_n(\xi)\big)d\xi$$

Take the limits for both sides as $n \to \infty$, we get

$$\lim_{n\to\infty} Hx_n(t) = \lim_{n\to\infty}\left(\frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^{\tau}\Phi\big(\xi, x_n(\xi)\big)d\xi\right] + \int_0^{t}\Phi\big(\xi, x_n(\xi)\big)d\xi\right)$$

The hypotheses (i)-(iii) allow for the utilization of the Lebesgue dominated convergence theorem [8] to interchange the limit and the integral. Then we can write that

$\lim_{n\to\infty}\Phi\big(\xi, x_n(\xi)\big) = \Phi\left(\xi, \lim_{n\to\infty} x_n(\xi)\right)$ with $x_n \to x$.

Which implies

$$\lim_{n\to\infty} \Phi(\xi, x_n(\xi)) = \Phi(\xi, x(\xi)),$$

and

$$\lim_{n\to\infty} Hx_n(t) = \frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^\tau \Phi(\xi, x(\xi))d\xi\right] + \int_0^t \Phi(\xi, x(\xi))d\xi = Hx(t)$$

Hence Hx is continuous, which means that H is continuous operator on Ω_r.

Therefore, the operator H has a fixed point on Ω_r(according to Schauder's Theorem [10]). Then, the integral equation (4) has at least one solution $x \in C[0,1]$.

We need to show that the integral equation (4) satisfies the nonlocal problem (1)-(2).

Differentiate the equation (4) with respect to t, we get

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^\tau \Phi(\xi, x(\xi))d\xi\right] + \int_0^t \Phi(\xi, x(\xi))d\xi\right) = \frac{d}{dt}\int_0^t \Phi(\xi, x(\xi))d\xi.$$

Since Φ is a measurable function in $t \in [0,1]$ and bounded by an integrable function $\mu(t)$, then $\Phi \in L^1[0,1]$ and

$$\frac{dx}{dt} = \Phi(t, x(t)) \quad a.e.\ t \in (0,1].$$

Put $t = \tau$ in equation (4), then multiply it by α, we get

$$\alpha\, x(\tau) = \frac{\alpha}{1+\alpha}\left[x_0 - \alpha\int_0^\tau \Phi(\xi, x(\xi))d\xi\right] + \alpha\int_0^\tau \Phi(\xi, x(\xi))d\xi. \qquad (7)$$

Put $t = 0$ in equation (4), we get

$$x(0) = \frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^\tau \Phi(\xi, x(\xi))d\xi\right]. \qquad (8)$$

By adding equations (7) and (8), we obtain $x(0) + \alpha x(\tau) = x_0$. Therefore, there exists an equivalence between the integral equation (4) and the nonlocal problem (1)-(2). Hence there exists at least one solution $x \in AC[0,1]$ to the nonlocal problem (1)-(2).

2.3. Uniqueness of the solution

To prove the uniqueness of the solution, we will consider the nonlocal problem (1)-(2) under the following hypotheses:

v) The function $\Phi: [0,1] \times \mathbb{R} \to \mathbb{R}$ is assumed to be Lebesgue measurable with respect to t for every $x \in \mathbb{R}$.

vi) There exists a positive constant L such that the function Φ satisfies the Lipschitz inequality
$|\Phi(t,x) - \Phi(t,y)| \le L|x - y|$.

vii) $\Phi(t,0) = \mu(t)$.

viii) $(1 + 2\alpha)L < (1 + \alpha)$

Theorem 2. Suppose that the hypotheses (v)-(viii) hold. Then, the uniqueness of the solution to the nonlocal problem (1)-(2) is established.

Proof.

In the light of the hypothesis (vi), the following expression is obtained.

$$|\Phi(t,x)| - |\Phi(t,0)| \leq |\Phi(t,x) - \Phi(t,0)| \leq a\,|x|.$$

Which implies that

$$|\Phi(t,x)| \leq |\Phi(t,0)| + a\,|x| \leq \mu(t) + a\,|x|.$$

Consequently, the validity of the hypothesis (ii) is confirmed. Therefore, due the theorem 1, the existence of at least one solution $x \in AC[0,1]$ to the nonlocal problem (1)-(2) is guaranteed. Let $x_1, x_2 \in AC[0,1]$ be two solutions of the nonlocal problem (1)-(2). Then

$$|x_1(t) - x_2(t)|$$

$$= \left|\frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^\tau \Phi(\xi, x_1(\xi))d\xi\right] + \int_0^t \Phi(\xi, x_1(\xi))d\xi - \left(\frac{1}{1+\alpha}\left[x_0 - \alpha\int_0^\tau \Phi(\xi, x_2(\xi))d\xi\right] + \int_0^t \Phi(\xi, x_2(\xi))d\xi\right)\right|$$

$$\leq \frac{\alpha}{1+\alpha}\int_0^\tau |\Phi(\xi, x_1(\xi)) - \Phi(\xi, x_2(\xi))|\,d\xi + \int_0^t |\Phi(\xi, x_1(\xi)) - \Phi(\xi, x_2(\xi))|\,d\xi.$$

$$|x_1(t) - x_2(t)| \leq \frac{\alpha}{1+\alpha} L\int_0^\tau |x_1(\xi) - x_2(\xi)|\,d\xi + L\int_0^t |x_1(\xi) - x_2(\xi)|\,d\xi.$$

$$|x_1(t) - x_2(t)| \leq \frac{\alpha}{1+\alpha} L\int_0^1 |x_1(\xi) - x_2(\xi)|\,d\xi + L\int_0^1 |x_1(\xi) - x_2(\xi)|\,d\xi.$$

By passing to the supermum norm, we obtain

$$\|x_1 - x_2\| \leq \left(\frac{\alpha}{1+\alpha} + 1\right) L\|x_1 - x_2\|$$

$$\left(1 - \frac{1+2\alpha}{1+\alpha}L\right)\|x_1 - x_2\| \leq 0$$

Since the condition $(1+2\alpha)L < (1+\alpha)$ holds true, then $\|x_1 - x_2\| = 0$. This leads to the conclusion that $x_1 = x_2$.

Which immediately yields that the integral equation (4) possesses one and only one solution $x \in AC[0,1]$. Therefore, the nonlocal problem (1)-(2) admits a unique solution.

Corollary 1. By setting $\alpha = 0$ in Theorem 1, the initial value problem

$$\frac{dx}{dt} = \Phi(t, x(t)), \quad t \in [0,1],$$

$$x(0) = x_0,$$

is shown to possess at least one solution $x \in AC[0,1]$, which is given by the expression

$$x(t) = x_0 + \int_0^1 \Phi(\xi, x(\xi))d\xi.$$

2.4. Continuous dependence of the solution

Now, we will prove Continuous dependence of the solution on the initial data x_0 and parameter α.

2.4.1. Continuous dependence of the solution on the initial data

Definition 1. We say that the solution of the nonlocal problem (1)-(2) depends continuously on x_0 if for all $\varepsilon > 0, \exists\, \delta > 0$ such that $|x_0 - x_0{}^*| \leq \delta$, then $\|x - x^*\| \leq \varepsilon$, when x^* is the unique solution of the nonlocal problem (1)-(2).

Theorem 3. Suppose that the hypotheses of theorem 2 are satisfied, then the solution to the nonlocal problem (1)-(2) exhibits continuous dependence on x_0.

Proof. Let x, x^* be the solutions of the nonlocal problem (1)-(2). Assume that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_0 - x_0{}^*| \leq \delta$. Then

$$\begin{aligned}
|x(t) - x^*(t)| &= \Bigg| \frac{1}{1+\alpha}\left[x_0 - \alpha \int_0^\tau \Phi\big(\xi, x(\xi)\big) d\xi\right] \\
&\quad + \int_0^t \Phi\big(\xi, x(\xi)\big) d\xi \\
&\quad - \left(\frac{1}{1+\alpha}\left[x_0{}^* - \alpha \int_0^\tau \Phi\big(\xi, x^*(\xi)\big) d\xi\right] + \int_0^t \Phi\big(\xi, x^*(\xi)\big) d\xi\right)\Bigg|, \\
&= \Bigg| \frac{1}{1+\alpha}(x_0 - x_0{}^*) \\
&\quad - \frac{\alpha}{1+\alpha}\int_0^\tau \left[\Phi\big(\xi, x(\xi)\big) - \Phi\big(\xi, x^*(\xi)\big)\right] d\xi \\
&\quad + \int_0^t \left[\Phi\big(\xi, x(\xi)\big) - \Phi\big(\xi, x^*(\xi)\big)\right] d\xi \Bigg|, \\
&\leq \frac{|x_0 - x_0{}^*|}{1+\alpha} \\
&\quad + \frac{\alpha}{1+\alpha}\int_0^\tau \left|\Phi\big(\xi, x(\xi)\big) - \Phi\big(\xi, x^*(\xi)\big)\right| d\xi \\
&\quad + \int_0^t \left|\Phi\big(\xi, x(\xi)\big) - \Phi\big(\xi, x^*(\xi)\big)\right| d\xi, \\
&\leq \frac{\delta}{(1+\alpha)} + \frac{\alpha}{1+\alpha} L \int_0^\tau |x(\xi) - x^*(\xi)|\, d\xi + L\int_0^t |x(\xi) - x^*(\xi)|\, d\xi, \\
&\leq \frac{\delta}{(1+\alpha)} + \frac{\alpha}{1+\alpha} L \int_0^1 |x(\xi) - x^*(\xi)|\, d\xi + L\int_0^1 |x(\xi) - x^*(\xi)|\, d\xi, \\
&\leq \frac{\delta}{(1+\alpha)} + \left(\frac{1+2\alpha}{1+\alpha}\right) L \int_0^1 |x(\xi) - x^*(\xi)|\, d\xi.
\end{aligned}$$

By passing to the supermum norm, we obtain

$$\|x - x^*\| \leq \frac{\delta}{(1+\alpha)} + \left(\frac{1+2\alpha}{1+\alpha}\right) L\, \|x - x^*\|.$$

Therefore, if $(1+2\alpha)L < (1+\alpha)$, we get

$$\|x - x^*\| \leq \frac{\delta}{(1+\alpha) - (1+2\alpha)L} \leq \varepsilon.$$

2.4.2. Continuous dependence of the solution on the parameter

Definition 2. We say that the solution of the nonlocal problem (1)-(2) depends continuously on α if for all $\varepsilon > 0, \exists\, \delta > 0$ such that $|\alpha - \alpha^*| \leq \delta$, then $\|x - x^*\| \leq \varepsilon$, when x^* is the unique solution of the nonlocal problem (1)-(2).

Theorem 4. Suppose that the hypotheses of theorem 2 are satisfied, then the solution to the nonlocal problem (1)-(2) exhibits continuous dependence on α.

Proof. Let x, x^* be the solutions of the nonlocal problem (1)-(2). Assume that for all $\varepsilon > 0$, there exist $\delta > 0$ such that $|\alpha - \alpha^*| \leq \delta$. Then

$$|x(t) - x^*(t)| = \left| \frac{1}{1+\alpha}\left[x_0 - \alpha \int_0^\tau \Phi(\xi, x(\xi))d\xi\right] + \int_0^t \Phi(\xi, x(\xi))d\xi - \left(\frac{1}{1+\alpha^*}\left[x_0 - \alpha^* \int_0^\tau \Phi(\xi, x^*(\xi))d\xi\right] + \int_0^t \Phi(\xi, x^*(\xi))d\xi\right)\right|$$

$$= \left| \frac{\alpha^* - \alpha}{(1+\alpha)(1+\alpha^*)} x_0 - \frac{\alpha}{1+\alpha}\int_0^\tau \Phi(\xi, x(\xi))d\xi + \frac{\alpha}{1+\alpha}\int_0^\tau \Phi(\xi, x^*(\xi))d\xi - \frac{\alpha}{1+\alpha}\int_0^\tau \Phi(\xi, x^*(\xi))d\xi + \frac{\alpha^*}{1+\alpha^*}\int_0^\tau \Phi(\xi, x^*(\xi))d\xi + \int_0^t [\Phi(\xi, x(\xi)) - \Phi(\xi, x^*(\xi))]d\xi \right|$$

$$\leq \frac{|\alpha^* - \alpha|}{(1+\alpha)(1+\alpha^*)}|x_0| + \frac{\alpha}{1+\alpha}\int_0^\tau |\Phi(\xi, x(\xi)) - \Phi(\xi, x^*(\xi))|d\xi + \frac{|\alpha^* - \alpha|}{(1+\alpha)(1+\alpha^*)}\int_0^\tau |\Phi(\xi, x^*(\xi))|d\xi + \int_0^t |\Phi(\xi, x(\xi)) - \Phi(\xi, x^*(\xi))|d\xi$$

$$\leq \frac{\delta}{(1+\alpha)(1+\alpha^*)}|x_0| + \frac{\alpha}{1+\alpha}L\int_0^\tau |x(\xi) - x^*(\xi)|\,d\xi + \frac{\delta}{(1+\alpha)(1+\alpha^*)}\int_0^\tau (\mu(\xi) + a\,|x^*|)\,d\xi + L\int_0^t |x(\xi) - x^*(\xi)|\,d\xi$$

$$\leq \frac{\delta}{(1+\alpha)(1+\alpha^*)}|x_0| + \frac{\alpha}{1+\alpha}L\int_0^1 |x(\xi) - x^*(\xi)|\,d\xi + \frac{\delta}{(1+\alpha)(1+\alpha^*)}\int_0^1 (\mu(\xi) + a\,|x^*|)\,d\xi + L\int_0^1 |x(\xi) - x^*(\xi)|\,d\xi$$

$$\leq \frac{\delta}{(1+\alpha)(1+\alpha^*)}|x_0| + \left(\frac{1+2\alpha}{1+\alpha}\right)L\,\|x - x^*\| + \frac{\delta}{(1+\alpha)(1+\alpha^*)}(\kappa + ar).$$

Since $x^* \in \Omega_r$, $|x^*| \leq r$ and $k = \int_0^1 \mu(\xi)d\xi$. Take $M = \kappa + ar \in \mathbb{R}$.

By passing to the supermum norm, we obtain

$$\|x - x^*\| \leq \frac{\delta}{(1+\alpha)(1+\alpha^*)}(|x_0| + M) + \left(\frac{1+2\alpha}{1+\alpha}\right)L\,\|x - x^*\|.$$

$$\|x - x^*\| \leq \frac{\delta(|x_0| + M)}{(1+\ \alpha)(1+\ \alpha^*)\left(1 - \frac{1+2\alpha}{1+\alpha}L\right)}.$$

Therefore, if $(1 + 2\alpha)L < (1 + \alpha)$, we get

$$\|x - x^*\| \leq \frac{\delta(|x_0| + M)}{(1+\ \alpha^*)(1 + \alpha - (1 + 2\alpha)L)} \leq \varepsilon.$$

2. Numerical Example.

Consider the following problem

$$\frac{dx}{dt} = \Phi\big(t, x(t)\big) = \frac{t}{1+t^2} + \frac{1}{4}|x(t)|\,,$$

$$0 \leq t \leq 1 \qquad (9)$$

Subject to the nonlocal condition:

$$x(0) + \tfrac{1}{2}\,x(1) = 5 \qquad (10)$$

Here $\alpha = \frac{1}{2}\,, \tau = 1$ and $x_0 = 5$.

To show existence of solutions to the problem (9)-(10).

We have

$$\left|\Phi\big(t, x(t)\big)\right| = \left|\frac{t}{1+t^2} + \frac{1}{4}|x(t)|\right| \leq \frac{t}{1+t^2} + \frac{1}{4}|x(t)|$$

Due to continuity of the function $\mu(t) = \frac{t}{1+t^2}$ on [0,1], we get $\mu \in L^1[0,1]$ with $\int_0^1 \mu(\xi)\,d\xi = \int_0^1 \frac{\xi}{1+\xi^2}\,d\xi = \frac{1}{2}\ln 2 = \kappa.$

Let $a = \frac{1}{4}$ which is a positive constant. The operator H is well-defined since $(1 + \alpha) = \frac{3}{2} \neq 0$.

Also,

$(1 + \alpha) = \frac{3}{2} \neq a(1 + 2\alpha) = \frac{1}{2}$.

Let the subset $\Omega_r = \{x: \|x\| \leq r\} \subset C[0,1]$ with $r = \frac{|x_0| + (1+2\alpha)\|\mu\|}{(1+\alpha) - a(1+2\alpha)} = 5.69 \sim 5.7$, which means that any ball with radius $r \geq 5.7$ is invariant under the operator H.

According to theorem 1 there exists at least one solution $x \in AC[0,1]$ to the problem (9)-(10).

To show uniqueness of the solution to the problem (9)-(10).

We need to verify the Lipschitz inequality as follows

$|\Phi(t,x) - \Phi(t,y)| \leq \frac{1}{4}\big||x| - |y|\big| \leq \frac{1}{4}|x - y|$, with Lipschitz constant $L = \frac{1}{4}$ and $\Phi(t, 0) = \frac{t}{1+t^2} = \mu(t)$.

Substituting the values of α and L into the inequality $(1 + 2\alpha)L < (1 + \alpha)$, we obtain $\frac{1}{2} < \frac{3}{2}$.

Therefore, due the theorem 2, the nonlocal problem (9)-(10) has exactly one solution $x \in AC[0,1]$.

To show continuous dependences of the solution on x_0, we will use **bvp4c** to solve boundary value problems in **MATLAB**, with two different values of x_0 and the parameter $\alpha = 0.5$ in both two cases. Then we will use the same tool in **MATLAB** to show continuous dependences of the solution on the parameter α with two different values of α and keep the same initial data x_0 in both two cases as following.

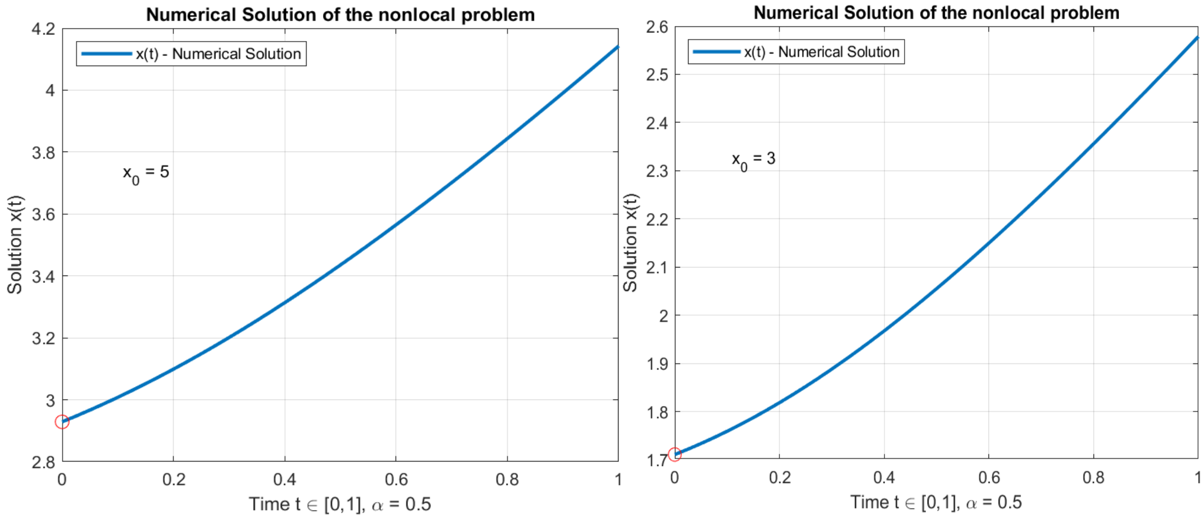

Figure1. Comparison curves of the solution for $x_0 = 5$ and for $x_0 = 3$ with the parameter $\alpha = 0.5$.

The figure 1 illustrates the comparison between two solution curves at two different values of the initial date x_0 while the parameter α remains constant $\alpha = 0.5$, we observe that the system exhibits a clear response when the initial value is increased from $x_0 = 3$ to $x_0 = 5$; the entire solution curve rises upwards. This rise reflects the system's flexibility in resetting the initial value $x(0)$ from (1.71 to 2.93 respectively) to satisfy the given condition $x(0) + \frac{1}{2}\, x(1) = x_0$, which reflecting the continuous sensitivity of the unique solution toward the initial data x_0.

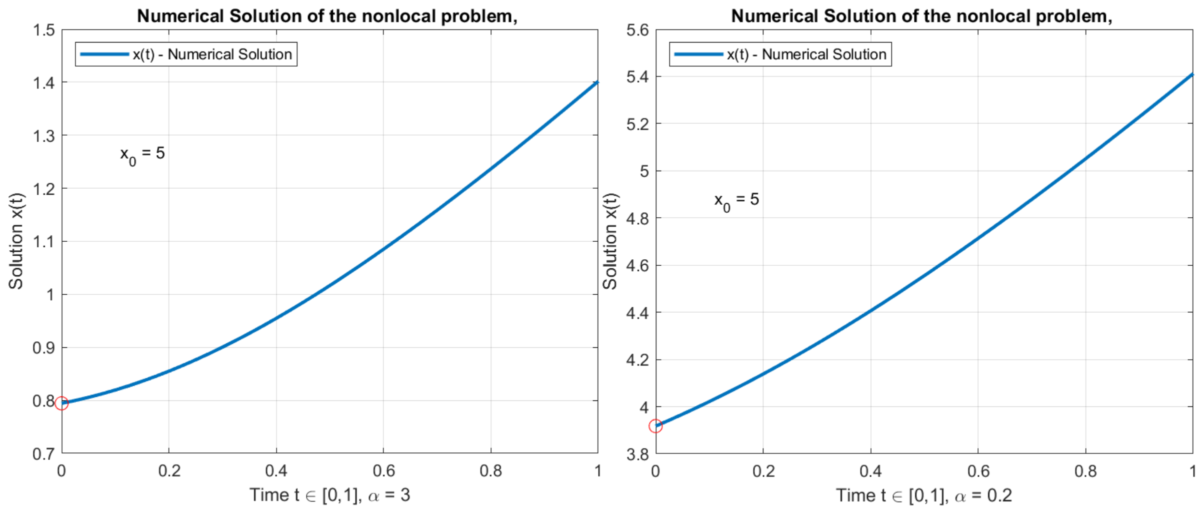

Figure2. Comparison curves of the solution for $\alpha = 3$ and $\alpha = 0.2$ with $x_0 = 5$.

The figure 2 illustrates the effect of the parameter α on the solution's behavior when the initial data $x_0 = 5$ remains constant. We observe that at $\alpha = 0.2$, the solution curve starts from a higher point, $x(0) = 3.92$. While conversely, as α increases; $\alpha = 3$, the entire curve shifts downwards, and the solution begins at a lower initial value $x(0) = 0.8$. We can say that, the parameter α determines the extent to which the ending value $x(1)$ affects the initial value $x(0)$. When $\alpha = 0.2$, the effect of the nonlocal condition on the integral equation (4) is minimal, as the effect of the nonlocal term $x(1)$ is very small ($\frac{\alpha}{1+\alpha} \approx 0.16$) on the initial value $x(0)$. However, when $\alpha = 3$, the ending value $x(1)$ exerts a significant influence on the initial value $x(0)$ ($\frac{\alpha}{1+\alpha} = 0.75$). Therefore, to satisfy the nonlocal condition $x(0) + \alpha\, x(1) = 5$, the only mathematical possibility is that the entire curve of $x(t)$ shifts downwards. This confirms the continuous sensitivity of the unique solution toward the parameter α.

3. Conclusions

The study concluded that the nonlinear differential problem with a one-parameter nonlocal condition possesses at least one solution when the appropriate assumptions related to measurability, continuity, and the linear growth condition of the nonlinear function are satisfied.

The results also showed that transforming the original differential problem into an equivalent integral equation is a fundamental step in the analysis, as it allows the use of fixed point theorems to prove the existence of a solution and facilitates the study of its mathematical properties.

The research proved that Schauder's Fixed Point Theorem provides a suitable framework for establishing the existence of the solution, after verifying that the defined operator on the appropriate function space is continuous and compact, and maps a closed convex subset into itself.

Furthermore, the study demonstrated that the Lipschitz condition, together with suitable restrictions on the constants and the parameter, is sufficient to guarantee the uniqueness of the solution. This means that the studied problem admits one and only one solution under the specified assumptions.

The research also proved that the solution depends continuously on the initial data. In other words, any small change in the initial data leads to a limited and regular change in the solution, which reflects the analytical stability of the problem.

In addition, the study established that the solution depends continuously on the parameter appearing in the nonlocal condition. This indicates that changes in the parameter do not cause sudden disturbances in the behavior of the solution, but rather affect it in a regular and traceable manner.

The numerical results obtained using MATLAB supported the theoretical findings. The graphs and numerical curves showed that the solution responds to variations in the initial data and the parameter in a manner consistent with the results of existence, uniqueness, and continuous dependence.

Accordingly, the study highlights the importance of combining theoretical analysis with numerical simulation in the investigation of nonlocal differential problems, especially in cases where explicit or direct solutions are difficult to obtain.

4. Recommendations

The study recommends extending the analysis to nonlinear differential equations of higher order or fractional differential equations equipped with nonlocal conditions, due to the importance of such models in representing more complex phenomena.

It is also recommended to study nonlocal problems involving more than one parameter, in order to analyze the effect of multiple parameters on the existence, uniqueness, and stability of solutions.

Future studies may also apply other numerical methods, such as the finite difference method, the finite element method, or iterative analytical methods, and compare their results with those obtained by MATLAB in the present study.

It is further recommended to investigate the numerical stability of solutions in greater depth by testing different values of the parameter and the initial data, and analyzing the sensitivity of the solution to these variations.
Moreover, future research may focus on applying the mathematical model to real problems in physics, engineering, or biological sciences, in order to emphasize the practical value of nonlocal differential equations.

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Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.